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Interaction-Flip Identities in Spin Glasses

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Abstract We study the properties of fluctuation for the free energies and internal energies of two spin glass systems that differ for having some set of interactions flipped. We show that their difference has a variance that grows like the volume of the flipped region. Using a new interpolation method, which extends to the entire circle the standard interpolation technique, we show by integration by parts that the bound imply new overlap identities for the equilibrium state. As a side result the case of the non-interacting random field is analyzed and the triviality of its overlap distribution proved.

Keywords Spin glasses · Interpolation · Identities · Fluctuation bounds

1 Introduction and Results

In this paper we investigate a new method to obtain overlap identities for the spin glass models. The strategy we use is the exploitation of a bound on the fluctuations of a quantity that compares a system with some Gaussian disorder with the system at a flipped ($J \to -J$) disorder. While the disordered averages are symmetric by interaction flip due to the symmetry of the distribution, the difference among them is an interesting random variable whose variance can be shown to grow at most like the volume (for extensive quantities).

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The identities are then deduced using some form of integration by parts in the same perspective in which they appear from stochastic stability [1] or the Ghirlanda Guerra method [7] in the mean field case or, more recently, for short range finite dimensional models [4, 5] (see also [2, 18]).

The interest of obtaining further information from the method of the identities lies on the fact that they provide a constraint for the overlap moments (or their distribution) and have the potential to reduce its degrees of freedom toward, possibly, a *mean field* structure like it is expected to have the Sherrington Kirkpatrick model.

More specifically the results of this paper consist of overlap identities for the quenched state which interpolate between a Gaussian spin glass and the system where the couplings in a subvolume (possibly coinciding with the whole volume) have been flipped. The interpolation is obtained by extending to the whole circle the Guerra Toninelli interpolation [9]. The bounds are derived from the concentration properties of the difference of the free energy per particle in the two settings, original and flipped.

As an example, one may consider the result which is stated in [12] (and quoted there as proved by Aizenman and Fisher) for the difference ΔF between the free energy of the Edwards-Anderson model on a d-dimensional lattice with linear size L and a volume L^d when going from periodic to antiperiodic boundary conditions on the hyperplane which is orthogonal to (say) the x-direction. The mentioned property is a bound for the variance of this quantity which grows no more than the volume of the hyperplane. Such an upper bound is equivalent to a bound for the stiffness exponent $\theta \leq (d-1)/2$ [3, 6, 17] (see also the discussion of that exponent in [20]). Although that bound is not expected to be saturated we prove here that it implies an identity for the equilibrium quantities. When expressed in terms of spin variables some of the overlap identities that we find generalize the structure of truncated correlation function that appear in [19] whose behaviour in the volume is related to the low temperature phase properties of the model. Consequences of our bound can also be seen at the level of the difference of internal energies. This second set of identities contains as a particular case some of the Ghirlanda-Guerra identities.

A quite interesting result, from the mathematical physics perspective, is provided by the analysis of the identities for the random field model without interaction. We show here that the new set of identities that we derive (and explicitly test) when considered together with the Ghirlanda Guerra ones provide a simple proof of triviality of the model i.e. the proof that the overlap is a non fluctuating quantity. We plan to apply the same method to the investigation of the random field model with ferromagnetic interactions.

The plan of the paper is the following. In the next section we define the setting of Gaussian spin glasses that we consider. Then in Sect. 3 we prove a lemma for the first two moments of the difference of free energies. This is obtained by studying a suitable interpolation on the circle for the linear combination of two independent Hamiltonians. Section 4 contains the proof of the concentration of measure results. The main results are given in Sects. 5 and 6, where the new overlap identities are stated. Finally in Sect. 7 we study the case of the random field model and shows how to derive the triviality of the model without making use of the explicit solution.

2 Definitions

We consider a disordered model of Ising configurations $\sigma_n = \pm 1, n \in \Lambda \subset \mathcal{L}$ for some subset Λ (volume $|\Lambda|$) of a lattice $\mathcal{L}$. We denote by Σ_Λ the set of all $\sigma = \{\sigma_n\}_{n\in\Lambda}$, and $|\Sigma_\Lambda| = 2^{|\Lambda|}$. In the sequel the following definitions will be used.

1. *Hamiltonian*. For every $\Lambda \subset \mathcal{L}$ let $\{H_\Lambda(\sigma)\}_{\sigma\in\Sigma_N}$ be a family of $2^{|\Lambda|}$ *translation invariant* (*in distribution*) *Gaussian* random variables defined, in analogy with [15], according to the general representation

$$H_\Lambda(\sigma) = -\sum_{X\subset\Lambda} J_X \sigma_X \tag{1}$$

where

$$\sigma_X = \prod_{i\in X} \sigma_i, \tag{2}$$

($\sigma_\emptyset = 0$) and the J's are independent Gaussian variables with mean

$$\mathrm{Av}(J_X) = 0, \tag{3}$$

and variance

$$\mathrm{Av}(J_X^2) = \Delta_X^2. \tag{4}$$

Given any subset $\Lambda' \subseteq \Lambda$, we also write

$$H_\Lambda(\sigma) = H_{\Lambda'}(\sigma) + H_{\Lambda\setminus\Lambda'}(\sigma) \tag{5}$$

where

$$H_{\Lambda'}(\sigma) = -\sum_{X\subset\Lambda'} J_X\sigma_X, \qquad H_{\Lambda\setminus\Lambda'}(\sigma) = -\sum_{\substack{X\subset\Lambda\\ X\not\subset\Lambda'}} J_X\sigma_X, \tag{6}$$

and

$$H_{\Lambda,\Lambda'}(\sigma) = -H_{\Lambda'}(\sigma) + H_{\Lambda\setminus\Lambda'}(\sigma) \tag{7}$$

will denote the Hamiltonian with the J couplings inside the region Λ' that have been flipped.

2. *Average and Covariance matrix*. The Hamiltonian $H_\Lambda(\sigma)$ has covariance matrix

$$\begin{aligned}\mathcal{C}_\Lambda(\sigma,\tau) &:= \mathrm{Av}\left(H_\Lambda(\sigma)H_\Lambda(\tau)\right)\\ &= \sum_{X\subset\Lambda} \Delta_X^2 \sigma_X\tau_X.\end{aligned} \tag{8}$$

By the Schwarz inequality

$$|\mathcal{C}_\Lambda(\sigma,\tau)| \le \sqrt{\mathcal{C}_\Lambda(\sigma,\sigma)}\sqrt{\mathcal{C}_\Lambda(\tau,\tau)} = \sum_{X\subset\Lambda}\Delta_X^2 \tag{9}$$

for all σ and τ.

3. *Thermodynamic Stability*. The Hamiltonian (1) is thermodynamically stable if there exists a constant $\bar{c}$ such that

$$\sup_{\Lambda\subset\mathcal{L}} \frac{1}{|\Lambda|}\sum_{X\subset\Lambda}\Delta_X^2 \le \bar{c} < \infty. \tag{10}$$

Thanks to the relation (9) a thermodynamically stable model fulfills the bound

$$\mathcal{C}_\Lambda(\sigma,\tau) \le \bar{c}|\Lambda| \tag{11}$$

and has an order 1 normalized covariance

$$c_\Lambda(\sigma,\tau) := \frac{1}{|\Lambda|}\mathcal{C}_\Lambda(\sigma,\tau). \tag{12}$$

4. *Random partition function.*

$$\mathcal{Z}_\Lambda(\beta) := \sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma)} \equiv \sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_{\Lambda'}(\sigma)-\beta H_{\Lambda\setminus\Lambda'}(\sigma)}, \tag{13}$$

$$\mathcal{Z}_{\Lambda,\Lambda'}(\beta) := \sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_{\Lambda,\Lambda'}(\sigma)} \equiv \sum_{\sigma\in\Sigma_\Lambda} e^{\beta H_{\Lambda'}(\sigma)-\beta H_{\Lambda\setminus\Lambda'}(\sigma)}. \tag{14}$$

5. *Random free energy/pressure.*

$$-\beta\mathcal{F}_\Lambda(\beta) := \mathcal{P}_\Lambda(\beta) := \ln\mathcal{Z}_\Lambda(\beta), \tag{15}$$

$$-\beta\mathcal{F}_{\Lambda,\Lambda'}(\beta) := \mathcal{P}_{\Lambda,\Lambda'}(\beta) := \ln\mathcal{Z}_{\Lambda,\Lambda'}(\beta). \tag{16}$$

6. *Random internal energy.*

$$\mathcal{U}_\Lambda(\beta) := \frac{\sum_{\sigma\in\Sigma_\Lambda} H_\Lambda(\sigma)e^{-\beta H_\Lambda(\sigma)}}{\sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma)}}, \tag{17}$$

$$\mathcal{U}_{\Lambda,\Lambda'}(\beta) := \frac{\sum_{\sigma\in\Sigma_\Lambda} H_{\Lambda,\Lambda'}(\sigma)e^{-\beta H_{\Lambda,\Lambda'}(\sigma)}}{\sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_{\Lambda,\Lambda'}(\sigma)}}. \tag{18}$$

7. *Quenched free energy/pressure.*

$$-\beta F_\Lambda(\beta) := P_\Lambda(\beta) := \mathrm{Av}(\mathcal{P}_\Lambda(\beta)), \tag{19}$$

$$-\beta F_{\Lambda,\Lambda'}(\beta) := P_{\Lambda,\Lambda'}(\beta) := \mathrm{Av}(\mathcal{P}_{\Lambda,\Lambda'}(\beta)). \tag{20}$$

8. *R-product random Boltzmann-Gibbs state.*

$$\Omega_\Lambda(-) := \sum_{\sigma^{(1)},\dots,\sigma^{(R)}} (-)\frac{e^{-\beta[H_\Lambda(\sigma^{(1)})+\cdots+H_\Lambda(\sigma^{(R)})]}}{[\mathcal{Z}_\Lambda(\beta)]^R}. \tag{21}$$

9. *Quenched equilibrium state.*

$$\langle -\rangle_\Lambda := \mathrm{Av}(\Omega_\Lambda(-)). \tag{22}$$

10. *Observables*. For any smooth bounded function $G(c_\Lambda)$ (without loss of generality we consider $|G|\le 1$ and no assumption of permutation invariance on G is made) of the covariance matrix entries we introduce the random (with respect to $\langle -\rangle$) $R\times R$ matrix of elements $\{q_{k,l}\}$ (called *generalized overlap*) by the formula

$$\langle G(q)\rangle := \mathrm{Av}(\Omega(G(c_\Lambda))). \tag{23}$$

E.g.: $G(c_\Lambda) = c_\Lambda(\sigma^{(1)},\sigma^{(2)})c_\Lambda(\sigma^{(2)},\sigma^{(3)})$

$$\langle q_{1,2}q_{2,3}\rangle = \mathrm{Av}\left(\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} c_\Lambda(\sigma^{(1)},\sigma^{(2)})c_\Lambda(\sigma^{(2)},\sigma^{(3)})\frac{e^{-\beta[\sum_{i=1}^{3} H_\Lambda(\sigma^{(i)})]}}{[\mathcal{Z}(\beta)]^3}\right). \tag{24}$$

Remark In the following, whenever there is no risk of confusion, the volume dependency in the quenched state or in the thermodynamic quantities will be dropped.

3 Preliminary: Interpolation on the Circle

Let $\xi = \{\xi_i\}_{1\le i\le n}$ and $\eta = \{\eta_i\}_{1\le i\le n}$ be two independent families of centered Gaussian random variables, each having covariance matrix $\mathcal{C}$, i.e.

$$\begin{aligned}\mathrm{Av}(\xi_i\xi_j) &= \mathcal{C}_{i,j},\\ \mathrm{Av}(\eta_i\eta_j) &= \mathcal{C}_{i,j},\\ \mathrm{Av}(\xi_i\eta_j) &= 0.\end{aligned}\tag{25}$$

Consider the following linear combination of ξ and η

$$x_i(t) = f(t)\xi_i + g(t)\eta_i$$

where the parameter $t \in [a,b] \subset \mathbb{R}$ and the two functions $f(t), g(t)$ take real values subject to the constraint

$$f(t)^2 + g(t)^2 = 1. \tag{26}$$

Chosing $f(t) = \cos(t)$, $g(t) = \sin(t)$ we obtain:

$$x_i(t) = \cos(t)\xi_i + \sin(t)\eta_i.$$

Because of the constraint (26), for any given time $t \in [a,b]$, the new centered Gaussian family $x(t) = \{x_i(t)\}_{1\le i\le n}$ has the same covariance structure of ξ and η:

$$\mathrm{Av}(x_i(t)x_j(t)) = \mathcal{C}_{i,j},$$

and hence the same distribution, independently of t (i.e. $x(t)$ is a stationary Gaussian process).

In the abstract set-up described above, we regard $x(t)$ as an interpolating Hamiltonian which is a linear combination of the random Hamiltonians ξ and η, with t-dependent weights that are the coordinates of a point on the circle of unit radius.[1] We introduce the interpolating random pressure:[2]

$$\mathcal{P}(t) = \ln Z(t) = \ln \sum_{i=1}^{n} e^{x_i(t)}, \tag{27}$$

and the notation $\langle C_{1,2}\rangle_{t,s}$ to denote the expectation of the covariance matrix in the deformed quenched state constructed from two independent copies with Boltzmann weights $x(t)$, respectively $x(s)$. Namely:

$$\langle C_{1,2}\rangle_{t,s} = \mathrm{Av}\sum_{i,j=1}^{n} \mathcal{C}_{i,j}\frac{e^{x_i(t)+x_j(s)}}{Z(t)Z(s)}. \tag{28}$$

[1] It is probably worth noting that any other parametrization of the unit circle would lead to the same expression as in (31).

[2] Here, in defining the interpolating (random) pressure, we absorb the temperature in the Hamiltonian.

The definition is extended in the obvious way to more than two copies. We will be interested in the random variable given by the difference of the pressures evaluated at the boundaries values

$$\mathcal{X}(a,b) = \mathcal{P}(b) - \mathcal{P}(a). \tag{29}$$

The following lemma gives an explicit expression for the first two moments of this random variable.

Lemma 1 *For the random variable $\mathcal{X}(a,b)$ defined above we have*

$$\mathrm{Av}(\mathcal{X}(a,b)) = 0 \tag{30}$$

and

$$\begin{aligned}\mathrm{Av}[(\mathcal{X}(a,b))^2] &= \int_a^b \int_a^b dt ds k_1(t,s) \langle C_{1,2}\rangle_{t,s} \\ &\quad - \int_a^b \int_a^b dt ds k_2(t,s) \left[\langle C_{1,2}^2\rangle_{t,s} - 2\langle C_{1,2}C_{2,3}\rangle_{s,t,s} + \langle C_{1,2}C_{3,4}\rangle_{t,s,s,t}\right]\end{aligned} \tag{31}$$

with

$$k_1(t,s) = \cos(t-s), \qquad k_2(t,s) = \sin^2(t-s). \tag{32}$$

Proof The stationarity of the Gaussian process $x(t)$ implies that $\mathrm{Av}(\mathcal{P}(t))$ is independent of t, this proves (30). As far as the computation of the second moment is concerned, starting from

$$\mathrm{Av}(\mathcal{X}(a,b)) = \int_a^b dt \mathrm{Av}(\mathcal{P}'(t)) = \int_a^b dt \sum_{i=1}^n \mathrm{Av}\left(x_i'(t)\frac{e^{x_i(t)}}{Z(t)}\right) \tag{33}$$

we have

$$\begin{aligned}\mathrm{Av}[(\mathcal{X}(a,b))^2] &= \int_a^b dt \int_a^b ds \mathrm{Av}(\mathcal{P}'(t)\mathcal{P}'(s)) \\ &= \int_a^b dt \int_a^b ds \sum_{i,j=1}^n \mathrm{Av}\left(x_i'(t)x_j'(s)\frac{e^{x_i(t)+x_j(s)}}{Z(t)Z(s)}\right).\end{aligned} \tag{34}$$

The computation of the average in the rightmost term of the previous formula, which is reported in Appendix A, gives

$$\begin{aligned}\mathrm{Av}\left(x_i'(t)x_j'(s)\frac{e^{x_i(t)+x_j(s)}}{Z(t)Z(s)}\right) &= \cos(t-s)\langle C_{1,2}\rangle_{t,s} \\ &\quad - \sin^2(s-t)\left(\langle C_{12}^2\rangle_{t,s} - 2\langle C_{12}C_{23}\rangle_{t,s,t} + \langle C_{12}C_{34}\rangle_{t,s,s,t}\right)\end{aligned} \tag{35}$$

proving (31). □

4 Bound on the Fluctuations of the Free Energy Difference

It is a well established fact that the random free energy per particles of Gaussian spin glasses satisfies concentration inequalities, implying in particular self-averaging. Here we prove that the same result holds for the variation in the random free energy (or equivalently the random pressure)

$$\mathcal{X}_{\Lambda,\Lambda'} = \mathcal{P}_{\Lambda} - \mathcal{P}_{\Lambda,\Lambda'} \tag{36}$$

induced by the change of the signs of the interaction in the subset $\Lambda' \subseteq \Lambda$. In general, the fact that the random free energy per particle concentrates around its mean as the system volume increases of the free energy can be obtained either by martingales arguments [5, 14] or by general Gaussian concentration of measure [10, 18]. Here we follow the second approach. Our formulation applies to both mean field and finite dimensional models and, for instance, includes the non summable interactions in finite dimensions [11] and the p-spin mean field model as well as the REM and GREM models.
Before stating the result, it is useful to notice that, as a consequence of the symmetry of the Gaussian distribution, the variation of the random pressure has a zero average:

$$\mathrm{Av}(\mathcal{X}_{\Lambda,\Lambda'}) = 0. \tag{37}$$

Lemma 2 *For every subset $\Lambda' \subset \Lambda$ the disorder fluctuation of the free energy variation $\mathcal{X}_{\Lambda,\Lambda'}$ satisfies the following inequality: for all $x > 0$*

$$\mathbb{P}\left(|\mathcal{X}_{\Lambda,\Lambda'}| \geq x\right) \leq 2\exp\left(-\frac{x^2}{8\pi\beta^2\bar{c}|\Lambda'|}\right) \tag{38}$$

with $\bar{c}$ the constant in the thermodynamic stability condition (cf. (10)). The variance of the free energy variation satisfies the bound

$$\mathrm{Var}(\mathcal{X}_{\Lambda,\Lambda'}) = \mathrm{Av}\left(\mathcal{X}^2_{\Lambda,\Lambda'}\right) \leq 16\pi\bar{c}\beta^2|\Lambda'|. \tag{39}$$

Proof Consider an $s > 0$ and let $x > 0$. By Markov inequality, one has

$$\begin{aligned}\mathbb{P}\left\{\mathcal{X}_{\Lambda,\Lambda'} \geq x\right\} &= \mathbb{P}\left\{\exp[s\mathcal{X}_{\Lambda,\Lambda'}] \geq \exp(sx)\right\}\\ &\leq \mathrm{Av}\left(\exp[s\mathcal{X}_{\Lambda,\Lambda'}]\right)\exp(-sx).\end{aligned} \tag{40}$$

To bound the generating function

$$\mathrm{Av}\left(\exp[s\mathcal{X}_{\Lambda,\Lambda'}]\right) \tag{41}$$

one introduces, for a parameter $t \in [0, \frac{\pi}{2}]$, the following interpolating partition functions:

$$Z^+(t) = \sum_{\sigma\in\Sigma_\Lambda} e^{-\beta\cos t H^{(1)}_{\Lambda'}(\sigma) - \beta H^{(3)}_{\Lambda\setminus\Lambda'}(\sigma) - \beta\sin t H^{(2)}_{\Lambda'}(\sigma)}, \tag{42}$$

$$Z^-(t) = \sum_{\sigma\in\Sigma_\Lambda} e^{\beta\cos t H^{(1)}_{\Lambda'}(\sigma) - \beta H^{(3)}_{\Lambda\setminus\Lambda'}(\sigma) + \beta\sin t H^{(2)}_{\Lambda'}(\sigma)}. \tag{43}$$

Here the Hamiltonians $H^{(1)}_{\Lambda'}(\sigma)$, $H^{(2)}_{\Lambda'}(\sigma)$, $H^{(3)}_{\Lambda\setminus\Lambda'}(\sigma)$, defined according to (6), depend on three independent copies $\{J^{(1)}_X\}_{X\subset\Lambda}, \{J^{(2)}_X\}_{X\subset\Lambda}, \{J^{(3)}_X\}_{X\subset\Lambda}$ of the Gaussian disorder characterized by (3), (4). Now we are ready to consider the interpolating function

$$\phi(t) = \ln \mathrm{Av}_3\mathrm{Av}_1 \left\{\exp\left(s\mathrm{Av}_2\left\{\ln\frac{Z^+(t)}{Z^-(t)}\right\}\right)\right\}, \tag{44}$$

where $\mathrm{Av}_1\{-\}$, $\mathrm{Av}_2\{-\}$ and $\mathrm{Av}_3\{-\}$ denote expectation with respect to the three independent families of Gaussian variables J_X. It is immediate to verify that

$$\phi(0) = \ln \mathrm{Av}\exp[s\mathcal{X}_{\Lambda,\Lambda'}], \tag{45}$$

and, using (37),

$$\phi\left(\frac{\pi}{2}\right) = 0. \tag{46}$$

This implies that

$$\mathrm{Av}\left(\exp[s\mathcal{X}_{\Lambda,\Lambda'}]\right) = e^{\phi(0)-\phi(\frac{\pi}{2})} = e^{-\int_0^{\frac{\pi}{2}}\phi'(t)dt}. \tag{47}$$

On the other hand, the function $\phi'(t)$ can be easily bounded. Defining

$$K(t) = \exp\left(s\mathrm{Av}_2\left\{\ln\frac{Z^+(t)}{Z^-(t)}\right\}\right) \tag{48}$$

the derivative is given by

$$\phi'(t) = \phi'_+(t) + \phi'_-(t) \tag{49}$$

where

$$\begin{aligned}\phi'_+(t) &= \frac{s\mathrm{Av}_3\mathrm{Av}_1\{K(t)\mathrm{Av}_2\{\frac{Z^+(t)'}{Z^+(t)}\}\}}{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\}},\\ \phi'_-(t) &= -\frac{s\mathrm{Av}_3\mathrm{Av}_1\{K(t)\mathrm{Av}_2\{\frac{Z^-(t)'}{Z^-(t)}\}\}}{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\}}.\end{aligned} \tag{50}$$

The first term in the derivative is

$$\phi'_+(t) = \frac{s\mathrm{Av}_3\mathrm{Av}_1\{K(t)\mathrm{Av}_2\{\sum_{\sigma\in\Sigma_\Lambda} p^+_t(\sigma)[\beta\sin t H^{(1)}_{\Lambda'}(\sigma) - \beta\cos t H^{(2)}_{\Lambda'}(\sigma)]\}\}}{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\}} \tag{51}$$

where

$$p^+_t(\sigma) = \frac{e^{-\beta\cos t H^{(1)}_{\Lambda'}(\sigma) - \beta H^{(3)}_{\Lambda\setminus\Lambda'}(\sigma) - \beta\sin t H^{(2)}_{\Lambda'}(\sigma)}}{Z^+(t)}. \tag{52}$$

Applying the integration by parts formula, a simple computation gives

$$\beta\sin t\mathrm{Av}_3\mathrm{Av}_1\left\{K(t)\mathrm{Av}_2\left\{\sum_\sigma p^+_t(\sigma)H^{(1)}_{\Lambda'}(\sigma)\right\}\right\}$$

$$= -s\beta^2 \sin t \cos t \,\mathrm{Av}_3\mathrm{Av}_1 \left\{ K(t) \sum_{X\subset\Lambda'} \Delta_X^2 [s_t^+(X)^2 + s_t^+(X)s_t^-(X)] \right\}$$

$$- \beta^2 \sin t \cos t \,\mathrm{Av}_3\mathrm{Av}_1 \left\{ K(t)\mathrm{Av}_2 \left\{ \sum_\sigma \mathcal{C}_{\Lambda'}(\sigma,\sigma) p_t^+(\sigma) \right\} \right\}$$

$$+ \beta^2 \sin t \cos t \,\mathrm{Av}_3\mathrm{Av}_1 \left\{ K(t)\mathrm{Av}_2 \left\{ \sum_{\sigma,\tau} \mathcal{C}_{\Lambda'}(\sigma,\tau) p_t^+(\sigma) p_t^+(\tau)) \right\} \right\} \tag{53}$$

and

$$-\beta \cos t \,\mathrm{Av}_3\mathrm{Av}_1 \left\{ K(t)\mathrm{Av}_2 \left\{ \sum_\sigma p^t(\sigma) H_{\Lambda'}^{(2)}(\sigma) \right\} \right\}$$

$$= \beta^2 \sin t \cos t \,\mathrm{Av}_3\mathrm{Av}_1 \left\{ K(t)\mathrm{Av}_2 \left\{ \sum_\sigma \mathcal{C}_{\Lambda'}(\sigma,\sigma) p_t^+(\sigma) \right\} \right\}$$

$$- \beta^2 \sin t \cos t \,\mathrm{Av}_1 \left\{ K(t)\mathrm{Av}_2 \left\{ \sum_{\sigma,\tau} \mathcal{C}_{\Lambda'}(\sigma,\tau) p_t^+(\sigma) p_t^+(\tau)) \right\} \right\} \tag{54}$$

where

$$s_t^+(X) = \mathrm{Av}_2 \left\{ \sum_\sigma \sigma_X p_t^+(\sigma) \right\}, \qquad s_t^-(X) = \mathrm{Av}_2 \left\{ \sum_\sigma \sigma_X p_t^-(\sigma) \right\} \tag{55}$$

and

$$p_t^-(\sigma) = \frac{e^{\beta \cos t H_{\Lambda'}^{(1)}(\sigma) - \beta H_{\Lambda\setminus\Lambda'}^{(3)}(\sigma) + \beta \sin t H_{\Lambda'}^{(2)}(\sigma)}}{Z^-(t)}. \tag{56}$$

Taking the difference between (53) and (54) one finds that

$$\phi_+'(t) = -s^2\beta^2 \sin t \cos t \frac{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\sum_{X\subset\Lambda'} \Delta_X^2 [s_t^+(X)^2 + s_t^+(X)s_t^-(X)]\}}{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\}}. \tag{57}$$

With a similar computation one obtains also

$$\phi_-'(t) = -s^2\beta^2 \sin t \cos t \frac{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\sum_{X\subset\Lambda'} \Delta_X^2 [s_t^-(X)^2 + s_t^+(X)s_t^-(X)]\}}{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\}}, \tag{58}$$

then we conclude that

$$\phi'(t) = -s^2\beta^2 \sin t \cos t \frac{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\sum_{X\subset\Lambda'} \Delta_X^2 [s_t^+(X) + s_t^-(X)]^2\}}{\mathrm{Av}_3\mathrm{Av}_1\{K(t)\}}. \tag{59}$$

Using the thermodynamic stability condition (11), this yields

$$|\phi'(t)| \le 4\beta^2 \bar{c} s^2 |\Lambda'| \tag{60}$$

from which it follows, using (47)

$$\mathrm{Av}(\exp[s\mathcal{X}_{\Lambda,\Lambda'}]) \le \exp(2\pi\beta^2 \bar{c} s^2 |\Lambda'|). \tag{61}$$

Inserting this bound into the inequality (40) and optimizing over s one finally obtains

$$\mathbb{P}(\mathcal{X}_{\Lambda,\Lambda'} \geq x) \leq \exp\left(-\frac{x^2}{8\pi\beta^2\bar{c}|\Lambda'|}\right). \tag{62}$$

The proof of inequality (38) is completed by observing that one can repeat a similar computation for $\mathbb{P}(\mathcal{X}_{\Lambda,\Lambda'} \leq -x)$. The result for the variance (39) is then immediately proved, thanks to (37), using the identity

$$\mathrm{Av}(\mathcal{X}^2_{\Lambda,\Lambda'}) = 2\int_0^\infty x\mathbb{P}(|\mathcal{X}_{\Lambda,\Lambda'}| \geq x)dx. \tag{63}$$

□

5 Overlap Identities from the Difference of Free Energy

We are now ready to state our first result.

Theorem 1 *Given a volume Λ, consider the Gaussian spin glass with Hamiltonian (1). For a subvolume $\Lambda' \subseteq \Lambda$ and a parameter $t \in [0,\pi]$, let*

$$\omega_t(-) = \sum_\sigma (-)e^{-H_\sigma(t)}/Z(t)$$

with

$$H_\sigma(t) = \cos(t)H^{(1)}_{\Lambda'}(\sigma) + \sin(t)H^{(2)}_{\Lambda'}(\sigma) + H_{\Lambda\setminus\Lambda'}(\sigma)$$

be the Boltzmann-Gibbs state which interpolates between the system with Gaussian disorder and the system with a flipped disorder in the region Λ' ($H^{(1)}_{\Lambda'}$ and $H^{(2)}_{\Lambda'}$ are two independent copies of the Hamiltonian in the subvolume Λ', $H_{\Lambda\setminus\Lambda'}(\sigma)$ is the Hamiltonian in the remaining part of the volume, they are all independent). Then, the following identities hold

$$\lim_{\Lambda,\Lambda'\nearrow\mathbb{Z}^d} \int_0^\pi\int_0^\pi dtds \sin^2(s-t)\Big[\langle(c^{\Lambda'}_{1,2})^2\rangle_{t,s} - 2\langle c^{\Lambda'}_{1,2}c^{\Lambda'}_{2,3}\rangle_{s,t,s} + \langle c^{\Lambda'}_{1,2}c^{\Lambda'}_{3,4}\rangle_{t,s,s,t}\Big] = 0 \tag{64}$$

where $\langle(c^{\Lambda'}_{1,2})^2\rangle_{t,s}$ (and analogously for the other terms) is the overlap of region $\Lambda' \subseteq \Lambda$ in the quenched state constructed form the interpolating Boltzmann-Gibbs state, e.g.

$$\langle(c^{\Lambda'}_{1,2})^2\rangle_{t,s} = \mathrm{Av}(\omega_t\omega_s(c^2_{\Lambda'}(\sigma,\tau))).$$

Proof The proof is obtained from a suitable combination of the results in the previous sections. For a parameter $t \in [0,\pi]$ we consider the interpolating random pressure

$$\mathcal{P}(t) = \ln\sum_{\sigma\in\Sigma_\Lambda} e^{x_\sigma(t)+H_{\Lambda\setminus\Lambda'}(\sigma)} \tag{65}$$

where

$$x_\sigma(t) = \cos(t)H^{(1)}_{\Lambda'}(\sigma) + \sin(t)H^{(2)}_{\Lambda'}(\sigma)$$

with $H^{(1)}_{\Lambda'}(\sigma), H^{(2)}_{\Lambda'}(\sigma)$ two independent copies of the Hamiltonian for the subvolume $\Lambda' \subseteq \Lambda$. The boundaries values give the random pressure of the original system when $t = 0$ and the random pressure of the system with the couplings J flipped on the subvolume Λ' when $t = \pi$, i.e.

$$\mathcal{P}(0) = \mathcal{P}_\Lambda,$$
$$\mathcal{P}(\pi) = \mathcal{P}_{\Lambda,\Lambda'}.$$

Application of Lemma 1 with $\xi_\sigma = H^{(1)}_{\Lambda'}(\sigma)$ and $\eta_\sigma = H^{(2)}_{\Lambda'}(\sigma)$ (the presence of the additional term $H_{\Lambda\setminus\Lambda'}(\sigma)$ in the random interpolating pressure does not change the result in the lemma, as far as the quenched state is correctly interpreted) gives

$$\begin{aligned}
&\mathrm{Var}(\mathcal{P}_\Lambda - \mathcal{P}_{\Lambda,\Lambda'})\\
&\quad = \Lambda' \int_0^\pi \int_0^\pi dt\,ds \cos(s-t) \langle c^{\Lambda'}_{1,2}\rangle_{t,s}\\
&\qquad + (\Lambda')^2 \int_0^\pi \int_0^\pi dt\,ds \sin^2(s-t) \Big[\langle (c^{\Lambda'}_{1,2})^2\rangle_{t,s} - 2\langle c^{\Lambda'}_{1,2} c^{\Lambda'}_{2,3}\rangle_{s,t,s} + \langle c^{\Lambda'}_{1,2} c^{\Lambda'}_{3,4}\rangle_{t,s,s,t}\Big].
\end{aligned} \tag{66}$$

On the other hand, Lemma 2 tell us that $\mathrm{Var}(\mathcal{P}_\Lambda - \mathcal{P}_{\Lambda,\Lambda'})$ is bounded above by a constant times the subvolume Λ'. As a consequence, the statement of the theorem follows. □

Remark When expressed in terms of the spin variables the polynomial in the integral (64) involves generalized truncated correlation functions. Indeed, for the model defined in Sect. 2, we have the following expressions

$$\begin{aligned}
\omega_{t,s}((C^{\Lambda'}_{1,2})^2) &= \sum_{X,Y\subset\Lambda'} \Delta^2_X \Delta^2_Y \omega_t(\sigma^{(1)}_X \sigma^{(1)}_Y)\omega_s(\sigma^{(2)}_X \sigma^{(2)}_Y),\\
\omega_{s,t,s}(C^{\Lambda'}_{1,2} C^{\Lambda'}_{2,3}) &= \sum_{X,Y\subset\Lambda'} \Delta^2_X \Delta^2_Y \omega_s(\sigma^{(1)}_X)\omega_t(\sigma^{(2)}_X \sigma^{(2)}_Y)\omega_s(\sigma^{(3)}_Y),\\
\omega_{t,s,s,t}(C^{\Lambda'}_{1,2} C^{\Lambda'}_{3,4}) &= \sum_{X,Y\subset\Lambda'} \Delta^2_X \Delta^2_Y \omega_t(\sigma^{(1)}_X)\omega_s(\sigma^{(2)}_X)\omega_s(\sigma^{(3)}_Y)\omega_t(\sigma^{(4)}_Y),
\end{aligned} \tag{67}$$

thus

$$\begin{aligned}
&\omega_{t,s}((c^{\Lambda'}_{1,2})^2) - 2\omega_{s,t,s}(c^{\Lambda'}_{1,2} c^{\Lambda'}_{2,3}) + \omega_{t,s,s,t}(c^{\Lambda'}_{1,2} c^{\Lambda'}_{3,4})\\
&\quad = \frac{1}{|\Lambda'|^2} \sum_{X,Y\subset\Lambda'} \Delta^2_X \Delta^2_Y [\omega_t(\sigma_X\sigma_Y) - \omega_t(\sigma_X)\omega_t(\sigma_Y)][\omega_s(\sigma_X\sigma_Y) - \omega_s(\sigma_X)\omega_s(\sigma_Y)]
\end{aligned} \tag{68}$$

where replica indices have been dropped. For the Edwards-Anderson model, which is obtained with $\Delta^2_X = 1$ if $X \in B' = \{(n,n') \in \Lambda' \times \Lambda', |n-n'| = 1\}$ and $\Delta^2_X = 0$ otherwise, the linear combination (68) of the moments of the link-overlap in the region Λ' is written in terms of truncated correlation functions, that is

$$\omega_{t,s}((c_{1,2}^{\Lambda'})^2) - 2\omega_{s,t,s}(c_{1,2}^{\Lambda'}c_{2,3}^{\Lambda'}) + \omega_{t,s,s,t}(c_{1,2}^{\Lambda'}c_{3,4}^{\Lambda'})$$
$$= \frac{1}{|\Lambda'|^2}\sum_{b,b'\in B'}\left[\omega_t(\sigma_b\sigma_{b'}) - \omega_t(\sigma_b)\omega_t(\sigma_{b'})\right]\left[\omega_s(\sigma_b\sigma_{b'}) - \omega_s(\sigma_b)\omega_s(\sigma_{b'})\right], \quad (69)$$

with $\sigma_b = \sigma_n\sigma'_n$, if $b = (n, n') \in B'$.

6 Overlap Identities from the Difference of Internal Energy

In this section we study the change in the internal energy after a flip of the couplings. We consider only the case of the flip of all the couplings in the entire volume.

Let us consider two centered Gaussian families $\xi = \{\xi_i\}_{1\le i\le n}$, $\eta = \{\eta_i\}_{1\le i\le n}$ with covariance structure given by

$$\mathrm{Av}\left(\xi_i\xi_j\right) = \mathrm{Av}\left(\eta_i\eta_j\right) = \mathcal{C}_{i,j} \quad (70)$$

with $\mathcal{C}_{i,i} = N$. We assume the thermodynamic stability condition to hold. It follows that N is proportional to the volume. For example, in the case of the Edwards-Anderson model on a d-dimensional lattice we would have $N = d|\Lambda|$. We introduce the random free energies

$$\mathcal{P}_\xi(\beta) = \ln Z_\xi(\beta) = \ln\sum_i e^{-\beta\xi_i}, \qquad \mathcal{P}_\eta(\beta) = \ln Z_\eta(\beta) = \ln\sum_i e^{-\beta\eta_i}, \quad (71)$$

with the random Boltzmann-Gibbs state $\omega_\xi(-)$, $\omega_\eta(-)$ and their quenched versions:

$$\langle -\rangle_\xi = \mathrm{Av}_\xi\omega_\xi(-), \qquad \langle -\rangle_\eta = \mathrm{Av}_\eta\omega_\eta(-). \quad (72)$$

With a slight abuse of notation we will use the previous symbols also to denote the product state acting on the replicated system. The free energy difference, obtained flipping the Hamiltonian η,

$$\mathcal{X}(\beta) = \mathcal{P}_\xi(\beta) - \mathcal{P}_{-\eta}(\beta) \equiv \ln\sum_i e^{-\beta\xi_i} - \ln\sum_i e^{\beta\eta_i}, \quad (73)$$

has a β-derivative given by the difference between the internal energies:

$$\mathcal{X}'(\beta) = -\omega_\xi(\xi) - \omega_{-\eta}(\eta). \quad (74)$$

Using the symmetry of the distribution of η, we have the identities[3]

$$\mathrm{Av}_\xi\omega_\xi(\xi) = -\beta(N - \mathrm{Av}_\xi\omega_\xi(C_{1,2})) = -\beta(N - \langle C_{1,2}\rangle_\xi), \quad (75)$$
$$\mathrm{Av}_\eta\omega_{-\eta}(\eta) = \beta(N - \mathrm{Av}_\eta\omega_{-\eta}(C_{1,2})) = \beta(N - \langle C_{1,2}\rangle_\eta). \quad (76)$$

[3] Indeed, from the symmetry of the Gaussian distribution, we have that for any function $f(\eta)$ the following equalities hold: $\mathrm{Av}_\eta f(\eta) = \mathrm{Av}_\eta f(-\eta) = \mathrm{Av}_{-\eta} f(-\eta)$. In particular if g is a function of the configurations of the replicated system, applying the previous remark to $f(\eta) = \omega_\eta(g)$ we obtain: $\langle g\rangle_\eta \equiv \mathrm{Av}_\eta\omega_\eta(g) = \mathrm{Av}_\eta\omega_{-\eta}(g) = \mathrm{Av}_{-\eta}\omega_{-\eta}(g) \equiv \langle g\rangle_{-\eta}$.

These properties will be tacitly used several time in this section.

The above formulae show that the disorder average of $\mathcal{X}'(\beta)$ vanishes

$$\mathrm{Av}_{\xi,\eta}(\mathcal{X}'(\beta)) = \beta(\langle C_{1,2}\rangle_\eta - \langle C_{1,2}\rangle_\xi) = 0, \tag{77}$$

since, obviously, $\langle C_{1,2}\rangle_\eta = \langle C_{1,2}\rangle_\xi$. Here $C_{1,2} = \{\mathcal{C}_{i,j}\}_{i,j}$ represents the covariance matrix whose entries are regarded as configurations of two replicas labeled 1 and 2. Thus, using the identity $\mathrm{Av}_\eta(\omega_{-\eta}(\eta)^2) = \mathrm{Av}_\xi(\omega_\xi(\xi)^2)$, we have that the variance of $\mathcal{X}'(\beta)$ is given by:

$$\mathrm{Av}_{\xi,\eta}(\mathcal{X}'(\beta)^2) = 2\mathrm{Av}_\xi(\omega_\xi(\xi)^2) + 2\mathrm{Av}_{\xi,\eta}(\omega_\xi(\xi)\omega_{-\eta}(\eta)). \tag{78}$$

Using the integration by parts formula, we obtain that

$$\mathrm{Av}_\xi(\omega_\xi(\xi)^2) = \mathrm{Av}_\xi \sum_{i,j} C_{i,j} \frac{e^{-\beta\xi_i-\beta\xi_j}}{Z_\xi(\beta)^2} + \mathrm{Av}_\xi \sum_{i,j}\sum_{k,\ell} C_{i,k}C_{j,\ell} \frac{\partial^2}{\partial\xi_\ell\partial\xi_k}\left[\frac{e^{-\beta\xi_i-\beta\xi_j}}{Z_\xi(\beta)^2}\right]. \tag{79}$$

The second term in the right-hand side of the previous formula requires a repeated application of the integration by parts formula, which gives:

$$\begin{aligned}\mathrm{Av}_\xi \sum_{i,j}\sum_{k,\ell} C_{i,k}C_{j,\ell} \frac{\partial^2}{\partial\xi_\ell\partial\xi_k}\left[\frac{e^{-\beta\xi_i-\beta\xi_j}}{Z_\xi(\beta)^2}\right] &= \beta^2 N(N-2\langle C_{1,2}\rangle_\xi) + \beta^2\langle C_{1,2}^2\rangle_\xi \\ &\quad - 6\beta^2\langle C_{1,2}C_{2,3}\rangle_\xi + 6\beta^2\langle C_{1,2}C_{3,4}\rangle_\xi.\end{aligned}$$

Since the first term in the right-hand side of (79) is quenched average of $C_{1,2}$, we conclude that

$$\begin{aligned}\mathrm{Av}_\xi(\omega_\xi(\xi)^2) &= \langle C_{1,2}\rangle + \beta^2 N(N-2\langle C_{1,2}\rangle) + \beta^2\langle C_{1,2}^2\rangle \\ &\quad - 6\beta^2\langle C_{1,2}C_{2,3}\rangle + 6\beta^2\langle C_{1,2}C_{3,4}\rangle\end{aligned} \tag{80}$$

dropping, here and in what follows, the unessential reference to ξ in the quenched averages. If the two families ξ and η were independent, then in (78) the average of the product would factorize $\mathrm{Av}_{\xi,\eta}(\omega_\xi(\xi)\omega_{-\eta}(\eta)) = -\beta^2(N-\langle C_{1,2}\rangle)^2$ giving:

$$\mathrm{Av}_{\xi,\eta}(\mathcal{X}'(\beta)^2) = 2\langle C_{1,2}\rangle + 2\beta^2\left(\langle C_{1,2}^2\rangle - \langle C_{1,2}\rangle^2\right) + 12\beta^2\left(\langle C_{1,2}C_{3,4}\rangle - \langle C_{1,2}C_{2,3}\rangle\right).$$

In this case the self averaging of the normalized quantity $\mathcal{X}'(\beta)^2/N$ (see Theorem 2) would lead, in the large volume limit $N\to\infty$, to the well known identity [8]

$$\langle c_{1,2}c_{2,3}\rangle - \langle c_{1,2}c_{3,4}\rangle = \frac{1}{6}\left(\langle c_{1,2}^2\rangle - \langle c_{1,2}\rangle^2\right). \tag{81}$$

However, our concern here is the computation of the quadratic fluctuations of $\mathcal{X}'(\beta)$ when the sign of a given Hamiltonian ξ is flipped in the whole volume. Therefore we have to set $\xi=\eta$ in (78). The computation requires, once again, the repeated use of the integration by parts formula

$$\mathrm{Av}_\xi(\omega_\xi(\xi)\omega_{-\xi}(\xi)) = \mathrm{Av}_\xi\left(\sum_{i,j}\xi_i\xi_j\frac{e^{-\beta\xi_i+\beta\xi_j}}{Z_\xi(\beta)Z_\xi(-\beta)}\right)$$
$$= \mathrm{Av}_\xi\sum_{i,j}C_{i,j}\frac{e^{-\beta\xi_i+\beta\xi_j}}{Z_\xi(\beta)Z_\xi(-\beta)}$$
$$+ \mathrm{Av}_\xi\sum_{i,j}\sum_{k,\ell}C_{i,k}C_{j,\ell}\frac{\partial^2}{\partial\xi_\ell\partial\xi_k}\left[\frac{e^{-\beta\xi_i+\beta\xi_j}}{Z_\xi(\beta)Z_\xi(-\beta)}\right]. \tag{82}$$

The average in (82) is expressed through a set of *mixed* quenched state: for instance, the first term in right-hand side of the previous equation is

$$\langle C_{1,2}\rangle_{+,-} = \mathrm{Av}_\xi\sum_{i,j}C_{i,j}\frac{e^{-\beta\xi_i+\beta\xi_j}}{Z_\xi(\beta)Z_\xi(-\beta)}. \tag{83}$$

Generalizing the previous definition we have, for instance, that $\langle - \rangle_{+,+,-,+}$ represents the thermal average taken with the usual Boltzmann factor (i.e. with the sign $-$ in the exponent) in the first, second and fourth copy, and with the opposite sign in the third one. Moreover, the symbol $\langle - \rangle_{+,+,+,\dots}$, with all the subscripts $+$ (or $-$, because of the symmetry of the Gaussian distribution), is the usual quenched measure $\langle - \rangle$. The explicit computation gives:

$$\frac{\partial^2}{\partial\xi_\ell\partial\xi_k}\left[\frac{e^{-\beta\xi_i+\beta\xi_j}}{Z_\xi(\beta)Z_\xi(-\beta)}\right] = -\beta^2N^2 + 2\beta^2N\langle C_{1,2}\rangle_{+,+} - \beta^2\langle C_{1,2}^2\rangle_{+,-} + 2\beta^2\langle C_{1,2}C_{2,3}\rangle_{+,-,+}$$
$$- 4\beta^2\langle C_{1,2}C_{2,3}\rangle_{+,+,-} + 4\beta^2\langle C_{1,2}C_{3,4}\rangle_{+,+,+,-}$$
$$- \beta^2\langle C_{1,2}C_{3,4}\rangle_{+,+,-,-} - \beta^2\langle C_{1,2}C_{3,4}\rangle_{+,-,+,-}$$

and finally:

$$\mathrm{Av}_\xi(\mathcal{X}'(\beta)^2) = 2(\langle C_{1,2}\rangle_{+,+} + \langle C_{1,2}\rangle_{+,-}) + 2\beta^2\left(\langle C_{1,2}^2\rangle_{+,+} - \langle C_{1,2}^2\rangle_{+,-}\right)$$
$$- 4\beta^2\left(3\langle C_{1,2}C_{2,3}\rangle_{+,+,+} - \langle C_{1,2}C_{2,3}\rangle_{+,-,+} + 2\langle C_{1,2}C_{2,3}\rangle_{+,+,-}\right)$$
$$+ 2\beta^2\left(6\langle C_{1,2}C_{3,4}\rangle_{+,+,+,+} + 4\langle C_{1,2}C_{3,4}\rangle_{+,+,+,-}\right.$$
$$\left.- \langle C_{1,2}C_{3,4}\rangle_{+,+,-,-} - \langle C_{1,2}C_{3,4}\rangle_{+,-,+,-}\right). \tag{84}$$

If we choose now ξ to be the Hamiltonian family defined in Sect. 2, we obtain the following:

Theorem 2 *Consider the Gaussian spin glass with Hamiltonian ξ given in* (1). *In the infinite volume limit and for almost all values of β, we have*

$$\left[\langle c_{1,2}^2\rangle_{+,+} - \langle c_{1,2}^2\rangle_{+,-}\right] - 2\left[3\langle c_{1,2}c_{2,3}\rangle_{+,+,+} - \langle c_{1,2}c_{2,3}\rangle_{+,-,+} + 2\langle c_{1,2}c_{2,3}\rangle_{+,+,-}\right]$$
$$+ \left[6\langle c_{1,2}c_{3,4}\rangle_{+,+,+,+} + 4\langle c_{1,2}c_{3,4}\rangle_{+,+,+,-} - \langle c_{1,2}c_{3,4}\rangle_{+,+,-,-} - \langle c_{1,2}c_{3,4}\rangle_{+,-,+,-}\right] = 0 \tag{85}$$

where $\langle c_{1,2}^2\rangle_{+,-}$ (and analogously for the other terms) is the overlap expectation in the quenched state constructed form the mixed Boltzmann-Gibbs state with one copy given by the original system and the other copy given by the flipped systems, e.g.

$$\langle c_{1,2}^2\rangle_{+,-} = \mathrm{Av}(\omega_\xi\omega_{-\xi}(c_\Lambda^2(\sigma,\tau))).$$

Proof The proof is a simple consequence of well known results. The sequence of convex functions $\mathcal{P}_\xi(\beta)/N$ converges almost everywhere in J to the limiting value $a(\beta)$ of its average and the convergence is self averaging (i.e. $\mathrm{Var}(\mathcal{P}_\xi(\beta)/N) \to 0$). By general convexity arguments [15] it follows that the sequence of derivatives $\mathcal{P}'_\xi(\beta)/N$ converges to $u(\beta) = a'(\beta)$ almost everywhere in β and also that the convergence is self averaging ($\mathrm{Var}(\mathcal{P}'_\xi(\beta)/N) \to 0$, β-a.e.) [13, 16]. These remarks apply obviously also to $\mathcal{P}_{-\xi}(\beta)/N$ and to its derivative, with the same limiting functions $a(\beta)$ and $a'(\beta)$. Thus we have that $\mathcal{X}(\beta)/N = \mathcal{P}_\xi(\beta)/N - \mathcal{P}_{-\xi}(\beta)/N$ and its derivative $\mathcal{X}'(\beta)/N$ vanish a.e. in J in the large volume limit. Moreover, $\mathrm{Var}(\mathcal{X}'(\beta)/N) = \mathrm{Var}(\mathcal{P}'_\xi(\beta)/N) + \mathrm{Var}(\mathcal{P}'_{-\xi}(\beta)/N) - 2\mathrm{cov}(\mathcal{P}'_\xi(\beta)/N, \mathcal{P}'_{-\xi}(\beta)/N)$, thus estimating the covariance with the Cauchy-Schwartz inequality we have

$$\mathrm{Var}(\mathcal{X}'(\beta)/N) \le 4\mathrm{Var}(\mathcal{P}'_\xi(\beta)/N) \to 0, \quad \beta\text{-a.e.} \tag{86}$$

for $N \to \infty$. Therefore, dividing (84) by N^2 and taking the limit we obtain the result. □

7 Triviality of the Random Field Model

In this section we compute explicitly the expression appearing in Theorem 1

$$\langle c_{1,2}^2\rangle_{t,s} - 2\langle c_{1,2}c_{2,3}\rangle_{s,t,s} + \langle c_{1,2}c_{3,4}\rangle_{t,s,s,t} \tag{87}$$

in the simple case of the random field. We will show that this linear combination of overlap moments vanishes pointwise for all values of t and s. We will then deduce the triviality of the order parameter for the random field model.

We consider two families J_i and $\tilde{J}_i$ for $i = 1, \dots, N$ of independent normally distributed centered random variables with variance 1:

$$\mathrm{Av}(J_i J_j) = \mathrm{Av}(\tilde{J}_i \tilde{J}_j) = \delta_{i,j}, \quad \mathrm{Av}(J_i \tilde{J}_j) = 0, \tag{88}$$

and the random field Hamiltonians

$$\xi_\sigma = \sum_{i=1}^N J_i \sigma_i, \qquad \eta_\sigma = \sum_{i=1}^N \tilde{J}_i \sigma_i, \tag{89}$$

where $\sigma_i = \pm 1$. We have that $\xi = \{\xi_\sigma\}_\sigma$ and $\eta = \{\eta_\sigma\}_\sigma$ are two independent centered Gaussian families (each having $n = 2^N$ elements indexed by the configurations σ, N being the volume) and covariance structure given by:

$$\begin{aligned} \mathrm{Av}(\xi_\sigma \xi_\tau) &\equiv \mathcal{C}_{\sigma,\tau} = Nq(\sigma,\tau), \\ \mathrm{Av}(\eta_\sigma \eta_\tau) &\equiv \mathcal{C}_{\sigma,\tau} = Nq(\sigma,\tau), \\ \mathrm{Av}(\xi_\sigma \eta_\tau) &= 0, \end{aligned} \tag{90}$$

where $q(\sigma,\tau)$ is the *site overlap* of the two configurations σ and τ:

$$q(\sigma,\tau) = \frac{1}{N}\sum_{i,j=1}^N \sigma_i \tau_j. \tag{91}$$

The interpolating Hamiltonian:

$$x_\sigma(t) = \cos(t)\xi_\sigma + \sin(t)\eta_\sigma, \tag{92}$$

which is a stationary Gaussian process with the same distribution of ξ and η:

$$\mathrm{Av}(x_\sigma(t)x_\tau(t)) = Nq(\sigma,\tau), \tag{93}$$

defines the quenched deformed state on the replicated system, whose averages are denoted with the usual notation, e.g. $\langle - \rangle_{t,s}$, $\langle - \rangle_{s,t,s}$, ...

Theorem 3 *Consider the random field spin glass with Hamiltonian* (89). *In the limit $N \to \infty$ and for all values of t and s we have*

$$\gamma_1\langle q_{1,2}^2\rangle_{t,s} + \gamma_2\langle q_{1,2}\rangle_{t,s}^2 + \gamma_3\langle q_{1,2}q_{2,3}\rangle_{s,t,s} + \gamma_4\langle q_{1,2}q_{3,4}\rangle_{t,s,s,t} = 0 \tag{94}$$

for any choice of real $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ with $\gamma_1+\gamma_2+\gamma_3+\gamma_4 = 0$.

Proof The simple proof relies on the following identities, derived in Appendix B:

$$\langle C_{1,2}\rangle_{t,s}^2 = \sum_{i=1}^N \Big(\mathrm{Av}\big(\tanh(G_i(t))\tanh(G_i(s))\big)\Big)^2 + \mathcal{Q}_N(t,s), \tag{95}$$

$$\langle C_{1,2}^2\rangle_{t,s} = 1 + \mathcal{Q}_N(t,s), \tag{96}$$

$$\langle C_{1,2}C_{2,3}\rangle_{s,t,s} = \sum_{i=1}^N \mathrm{Av}\left(\tanh^2(G_i(s))\right) + \mathcal{Q}_N(t,s), \tag{97}$$

$$\langle C_{12}C_{34}\rangle_{t,s,s,t} = \sum_{j=1}^N \mathrm{Av}\left(\tanh^2(G_j(t))\tanh^2(G_j(s))\right) + \mathcal{Q}_N(t,s), \tag{98}$$

where

$$G_i(t) = \cos(t)J_i + \sin(t)\tilde{J}_i, \tag{99}$$

and $\mathcal{Q}_N(t,s)$ is a term of order N^2, see (118). Thus:

$$\begin{aligned}
&\gamma_1\langle C_{1,2}^2\rangle_{t,s} + \gamma_2\langle C_{1,2}\rangle_{t,s}^2 + \gamma_3\langle C_{1,2}C_{2,3}\rangle_{s,t,s} + \gamma_3\langle C_{1,2}C_{3,4}\rangle_{t,s,s,t} \\
&\quad = \gamma_1 + \gamma_2\sum_{i=1}^N \Big(\mathrm{Av}\big(\tanh(G_i(t))\tanh(G_i(s))\big)\Big)^2 + \gamma_3\sum_{i=1}^N \mathrm{Av}\left(\tanh^2(G_i(s))\right) \\
&\qquad + \gamma_4\sum_{j=1}^N \mathrm{Av}\left(\tanh^2(G_j(t))\tanh^2(G_j(s))\right) \\
&\qquad + (\gamma_1+\gamma_2+\gamma_3+\gamma_4)\mathcal{Q}_N(t,s),
\end{aligned}$$

i.e. the linear combination of the covariance matrix moments is of order N. Thus, since $|\tanh(x)| < 1$, we have

$$\begin{aligned}
&\left|\gamma_1\langle C_{1,2}^2\rangle_{t,s} + \gamma_2\langle C_{1,2}\rangle_{t,s}^2 + \gamma_3\langle C_{1,2}C_{2,3}\rangle_{s,t,s} + \gamma_4\langle C_{1,2}C_{3,4}\rangle_{t,s,s,t}\right| \\
&\quad \le |\gamma_1| + (|\gamma_2| + |\gamma_3| + |\gamma_4|)N,
\end{aligned}$$

which can be rewritten, using the overlaps $q_{1,2}, q_{2,3}, q_{3,4}$ between replicas, as

$$\left|\gamma_1\langle q_{1,2}^2\rangle_{t,s} + \gamma_2\langle q_{1,2}\rangle_{t,s}^2 + \gamma_3\langle q_{1,2}q_{2,3}\rangle_{s,t,s} + \gamma_4\langle q_{1,2}q_{3,4}\rangle_{t,s,s,t}\right| \leq \frac{|\gamma_2|+|\gamma_3|+|\gamma_4|}{N} + \frac{|\gamma_1|}{N^2}. \tag{100}$$

□

Among the relations of Theorem 3, in the thermodynamic limit, we find the identity of Theorem 1 for the values $\gamma_1 = 1, \gamma_2 = 0, \gamma_3 = -2, \gamma_4 = 1$:

$$\langle q_{1,2}^2\rangle_{t,s} - 2\langle q_{1,2}q_{2,3}\rangle_{s,t,s} + \langle q_{1,2}q_{3,4}\rangle_{t,s,s,t} = 0 \tag{101}$$

and the Ghirlanda-Guerra identities: for $\gamma_1 = 1, \gamma_2 = 1, \gamma_3 = -2, \gamma_4 = 0$ we find

$$\langle q_{1,2}q_{2,3}\rangle_{s,t,s} = \frac{1}{2}\langle q_{1,2}^2\rangle_{t,s} + \frac{1}{2}\langle q_{1,2}\rangle_{t,s}^2; \tag{102}$$

for $\gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 0, \gamma_4 = -3$ we find

$$\langle q_{1,2}q_{3,4}\rangle_{s,t,s} = \frac{1}{3}\langle q_{1,2}^2\rangle_{t,s} + \frac{2}{3}\langle q_{1,2}\rangle_{t,s}^2. \tag{103}$$

Using (102) and (103) we can express (101) as:

$$\langle q_{1,2}^2\rangle_{t,s} - 2\langle q_{1,2}q_{2,3}\rangle_{s,t,s} + \langle q_{1,2}q_{3,4}\rangle_{t,s,s,t} = \frac{1}{3}(\langle q_{1,2}^2\rangle_{t,s} - \langle q_{1,2}\rangle_{t,s}^2). \tag{104}$$

The identity derived from the flip of the coupling thus imply a trivial order parameter distribution. Indeed, since the identity (101) is true for every t and s we can choose $t = s = 0$ and then the interpolating states reduce to the usual quenched Boltzmann-Gibbs state. From (104) we deduce a trivial overlap distribution.

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Appendix A

In this appendix we will use the Gaussian integration by parts formula for correlated Gaussian random variables $z_1, \dots, z_n$:

$$\mathrm{Av}(z_j\psi(z_1,\dots,z_n)) = \sum_{i=1}^{n} \mathrm{Av}\left(z_j z_i\right) \mathrm{Av}\left(\frac{\partial\psi(z_1,\dots,z_n)}{\partial z_i}\right), \tag{105}$$

to compute the second moment of the pressure difference $\mathcal{X}(a,b)$. We have to evaluate the average inside the integral (34)

$$
\begin{aligned}
\sum_{i,j=1}^{n} \mathrm{Av}\left(x_i'(t)x_j'(s)B(i,j;t,s)\right) \\
&= \sin(t)\sin(s)\sum_{i,j=1}^{n} \mathrm{Av}\left(\xi_i\xi_j B(i,j;t,s)\right) \\
&\quad - \sin(t)\cos(s)\sum_{i,j=1}^{n} \mathrm{Av}\left(\xi_i\eta_j B(i,j;t,s)\right) \\
&\quad - \sin(s)\cos(t)\sum_{i,j=1}^{n} \mathrm{Av}\left(\xi_j\eta_i B(i,j;t,s)\right) \\
&\quad + \cos(t)\cos(s)\sum_{i,j=1}^{n} \mathrm{Av}\left(\eta_i\eta_j B(i,j;t,s)\right),
\end{aligned}
\tag{106}
$$

where, for the sake of notation, we have introduced the symbol

$$
B(i,j;t,s) = \frac{e^{x_i(t)+x_j(s)}}{Z(t)Z(s)}.
$$

Applying (105) twice, we obtain

$$
\begin{aligned}
\mathrm{Av}\left(\xi_i\xi_j B(i,j;t,s)\right) &= \mathcal{C}_{i,j}\mathrm{Av}\left(B(i,j;t,s)\right) \\
&\quad + \sum_{k,\ell=1}^{n} \mathcal{C}_{i,k}\mathcal{C}_{j,\ell}\mathrm{Av}\left(\frac{\partial^2}{\partial\xi_k\xi_\ell}B(i,j;t,s)\right),
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
\mathrm{Av}\left(\eta_i\eta_j B(i,j;t,s)\right) &= \mathcal{C}_{i,j}\mathrm{Av}\left(B(i,j;t,s)\right) \\
&\quad + \sum_{k,\ell=1}^{n} \mathcal{C}_{i,k}\mathcal{C}_{j,\ell}\mathrm{Av}\left(\frac{\partial^2}{\partial\eta_\ell\eta_k}B(i,j;t,s)\right),
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
\mathrm{Av}\left(\xi_i\eta_j B(i,j;t,s)\right) &= \mathrm{Av}\left(\xi_j\eta_i B(i,j;t,s)\right) \\
&= \sum_{k,\ell=1}^{n} \mathcal{C}_{i,k}\mathcal{C}_{j,\ell}\mathrm{Av}\left(\frac{\partial^2}{\partial\xi_k\eta_\ell}B(i,j;t,s)\right).
\end{aligned}
\tag{109}
$$

The combination of the first two terms in the right-hand sides of (107) and (108) with the trigonometric coefficients given by (106) produce the quenched expectation $\cos(t-s)\langle C_{1,2}\rangle_{t,s}$.
The explicit computation of the derivatives is long but not difficult; the result is:

$$
\frac{\partial^2}{\partial\xi_k\partial\xi_\ell}B(i,j;t,s) = B(i,j;t,s)\left\{\cos^2(t)A_1 + \cos^2(s)A_2 + \cos(t)\cos(s)(A_3+A_4)\right\},
$$

$$
\frac{\partial^2}{\partial\eta_k\partial\eta_\ell}B(i,j;t,s) = B(i,j;t,s)\left\{\sin^2(s)A_1 + \sin^2(s)A_2 + \sin(t)\sin(s)(A_3+A_4)\right\},
$$

$$\frac{\partial^2}{\partial \xi_k \eta_\ell} B(i,j;t,s) = B(i,j;t,s)\,\{\sin(t)\cos(t)A_1 + \sin(s)\cos(s)A_2 + \sin(t)\cos(s)A_3 + \sin(s)\cos(t)A_4\},$$

where A_1, A_2, A_3, A_4 are combinations of Kronecker delta functions depending on the indices i, j, ℓ, k and Boltzmann weights for the hamiltonians $x(t)$ and $x(s)$.

Using the previous formulas for the second derivatives and formulas (107), (108) and (109), we see that the right-hand side of (106) contains a linear combination of functions A_j with trigonometric coefficients given by the product of four factors taken from $\{\cos(t), \sin(t), \cos(s), \sin(s)\}$. It is not difficult to recognize that the coefficient of A_3 is $-\sin^2(s-t)$ while the other are zero. Thus,

$$\sum_{i,j=1}^{n} \mathrm{Av}\left(x_i'(t)x_j'(s)B(i,j;t,s)\right) = \cos(t-s)\langle C_{1,2}\rangle_{t,s} - \sin^2(s-t)\mathrm{Av}\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \mathcal{C}_{i,k}\mathcal{C}_{j,\ell}A_3 B(i,j;t,s)$$

and since

$$A_3 = \delta_{\ell,i}\delta_{k,j} - \delta_{\ell,i}\frac{e^{x_k(s)}}{Z(s)} - \delta_{k,j}\frac{e^{x_\ell(t)}}{Z(t)} + \frac{e^{x_\ell(t)}}{Z(t)}\frac{e^{x_k(s)}}{Z(s)}$$

we obtain

$$\mathrm{Av}\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n} \mathcal{C}_{i,k}\mathcal{C}_{j,\ell}A_3 B(i,j;t,s) = \langle C_{12}^2\rangle_{t,s} - 2\langle C_{12}C_{23}\rangle_{t,s,t} + \langle C_{12}C_{34}\rangle_{t,s,s,t}$$

which proves (35).

Appendix B

In this appendix we prove the identities (95), (96), (97), (98). Recalling the definition (99) of the Gaussian variables $G_i(t)$, we can define the interpolating partition function

$$Z(t) = \sum_\sigma \exp(x_\sigma(t)) = \sum_\sigma \exp\left(\sum_{i=1}^{N} G_i(t)\sigma_i\right). \tag{110}$$

A simple computation shows that

$$Z(t) = 2^N \prod_{i=1}^{N} \cosh G_i(t). \tag{111}$$

For any integer $M < N$, we consider the sublattice $\Lambda_M = \{N-M+1, \dots, N\} \subseteq \Lambda_N \equiv \{1, \dots, N\}$ with its spin-configuration space $S_M = \{-1, 1\}^{\Lambda_M}$ and the subspace $S_M^+ = \{(+1, \sigma_{N-M+2}, \dots, \sigma_N), \sigma_i = \pm 1\}$ (we will drop the subscript when it is equal to N, e.g.

$S \equiv S_N$, $S^+ \equiv S_N^+$, etc.). The interpolating Boltzmann-Gibbs random state on the lattice Λ_M is

$$\omega_t^M(f) = \frac{\sum_{\sigma \in S_M} f(\sigma) \exp(\sum_{i=N-M+1}^N G_i(t)\sigma_i)}{Z_M(t)},$$
$$Z_M(t) = \sum_{\sigma \in S_M} \exp\left(\sum_{i=N-M+1}^N G_i(t)\sigma_i\right), \tag{112}$$

where f is a function on S_M. This definition extends in the obvious way to the R-fold product; for instance the 2-product measure for the parameter values t and s is given by

$$\omega_{t,s}^M(f) = \frac{\sum_{\sigma,\tau \in S_M} f(\sigma,\tau) \exp(\sum_{i=N-M+1}^N G_i(t)\sigma_i + \sum_{i=N-M+1}^N G_i(s)\tau_i)}{Z_M(t) Z_M(s)} \tag{113}$$

where f is a function on $S_M \times S_M$. In the sequel we will also write $\omega_{t,s}(-)$ instead of $\omega_{t,s}^N(-)$.

The computation of the moments of the covariance matrix is done evaluating (by induction on M) the averages of the products of the overlaps between configurations of S_M

$$q_M(\sigma,\tau) = \frac{1}{M} \sum_{i=N-M+1}^N \sigma_i \tau_i. \tag{114}$$

Indeed, the explicit computation shows that:

$$\omega_{t,s}^N(q_N) = \frac{1}{N} \tanh G_1(t) \tanh G_1(s) + \frac{N-1}{N} \omega_{t,s}^{N-1}(q_{N-1}), \tag{115}$$

then iterating the previous formula $N-1$ times we obtain

$$\omega_{t,s}^N(q_N) = \frac{1}{N} \sum_{j=1}^N \tanh(G_j(t)) \tanh(G_j(s)) \tag{116}$$

since $\omega_{t,s}^1(q_1) = \tanh(G_N(t)) \tanh(G_N(s))$. Recalling the relation between overlaps and covariance (90) and taking the average with respect to the disorder, we obtain:

$$\langle C_{1,2} \rangle_{t,s} = \sum_{i=1}^N \mathrm{Av}\big(\tanh(G_i(t)) \tanh(G_i(s))\big), \tag{117}$$

thus

$$\langle C_{1,2} \rangle_{t,s}^2 = \sum_{i=1}^N \Big(\mathrm{Av}\big(\tanh(G_i(t)) \tanh(G_i(s))\big)\Big)^2 + \mathcal{Q}_N(t,s)$$

where

$$\mathcal{Q}_N(t,s) = 2 \sum_{1 \le j < \ell \le N} \mathrm{Av}[\tanh(G_j(t)) \tanh(G_j(s))] \times \mathrm{Av}[\tanh(G_\ell(t)) \tanh(G_\ell(s))] \tag{118}$$

is a term of order N^2. This proves (95).
For the squared overlap the following relation holds

$$\omega_{t,s}(q_N^2) = \frac{1}{N} + \frac{2}{N^2}\sum_{j=1}^{N-1}(N-j)\tanh(G_j(t))\tanh(G_j(s))\omega_{t,s}^{N-j}(q_{N-1}). \qquad (119)$$

Since for $M \leq N$

$$\omega_{t,s}^{M}(q_M) = \frac{1}{M}\sum_{j=N-M+1}^{N}\tanh(G_j(t))\tanh(G_j(s)) \qquad (120)$$

we can write

$$\omega_{t,s}(q_N^2) = \frac{1}{N^2} + \frac{2}{N^2} \times \sum_{1\leq j<\ell\leq N}\tanh(G_j(t))\tanh(G_j(s))\tanh(G_\ell(t))\tanh(G_\ell(s)) \qquad (121)$$

and finally

$$\langle C_{1,2}^2\rangle_{t,s} = 1 + 2\sum_{1\leq j<\ell\leq N}\mathrm{Av}\left[\tanh(G_j(t))\tanh(G_j(s))\tanh(G_\ell(t))\tanh(G_\ell(s))\right].$$

From the independence of the random variables $G_i(t)$ (see (99)), we have that the average in the right hand side of the previous formula factorizes, thus we obtain (96):

$$\langle C_{1,2}^2\rangle_{t,s} = 1 + \mathcal{Q}_N(t,s).$$

The second term in (87) is computed considering the average of $q_N(\sigma,\gamma)q_N(\gamma,\tau)$ where $\gamma,\sigma,\tau \in S$. We have

$$\begin{aligned}\omega_{s,t,s}^{N}(q_N(\sigma,\gamma)q_N(\gamma,\tau)) = {} & \frac{1}{N^2}\tanh^2(G_1(s)) \\ & + \frac{2}{N^2}\tanh(G_1(t))\tanh(G_1(s))\sum_{j=2}^{N}\tanh(G_j(t))\tanh(G_j(s)) \\ & + \left(\frac{N-1}{N}\right)^2\omega_{s,t,s}^{N-1}(q_{N-1}(\sigma',\gamma')q_{N-1}(\gamma',\tau')),\end{aligned} \qquad (122)$$

where σ',γ',τ' are the restriction of σ,γ,τ to S_{N-1}. As in the previous cases, iterating this formula and taking into account that $\omega_{s,t,s}^{1}(q_1(\sigma,\gamma)q_1(\gamma,\tau)) = \tanh^2(G_N(s))$ we obtain

$$\begin{aligned}\omega_{s,t,s}^{N}(q_N(\sigma,\gamma)q_N(\gamma,\tau)) = {} & \frac{1}{N^2}\sum_{i=1}^{N}\tanh^2(G_i(s)) + \frac{2}{N^2}\sum_{j=1}^{N-1}\tanh(G_j(t))\tanh(G_j(s)) \\ & \times \sum_{\ell=j+1}^{N}\tanh(G_\ell(t))\tanh(G_\ell(s)),\end{aligned} \qquad (123)$$

then

$$\langle C_{1,2}C_{2,3}\rangle_{s,t,s} = \sum_{i=1}^{N} \mathrm{Av}\left(\tanh^2(G_i(s))\right) + \mathcal{Q}_N(t,s),$$

which proves (97). The computation of the last term in (87) is simple because in this case the random product measure factorizes:

$$\omega_{t,s,s,t}(q(\sigma,\tau)q(\gamma,\kappa)) = \omega_{t,s}(q(\sigma,\tau))\omega_{s,t}(q(\gamma,\kappa)). \tag{124}$$

Then, using (116) we have

$$\omega_{t,s,s,t}(q(\sigma,\tau)q(\gamma,\kappa)) = \frac{1}{N^2}\sum_{i,j=1}^{N} \tanh(G_j(t))\tanh(G_j(s))\tanh(G_i(t))\tanh(G_i(s)) \tag{125}$$

and

$$\langle C_{12}C_{34}\rangle_{t,s,s,t} = \sum_{i,j=1}^{N} \mathrm{Av}\left(\tanh(G_j(t))\tanh(G_j(s))\tanh(G_i(t))\tanh(G_i(s))\right) \tag{126}$$

which, using the symmetry of $a_{i,j} = \mathrm{Av}(\tanh(G_j(t))\tanh(G_j(s))\tanh(G_i(t))\tanh(G_i(s)))$, gives (98):

$$\langle C_{12}C_{34}\rangle_{t,s,s,t} = \sum_{j=1}^{N} \mathrm{Av}\left(\tanh^2(G_j(t))\tanh^2(G_j(s))\right) + \mathcal{Q}_N(t,s).$$

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